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A NOTE ON THE CLASS GROUP OF SURFACES IN 3-SPACE

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Let A be a regular factorial ring which is a ring of quotients of some 3-dimensional finitely generated $\mathbb{C}$-algebra. The examples we are keeping in mind are $A=\mathbb{C}[t_1,t_2,t_3]=$ polynomial ring in 3 variables and $A=\mathcal{O}_{x,X}=$ local ring of a smooth closed point x on a complex algebraic 3-fold X.

Now given a height-2 prime $q\subset A$ there does not exist in general a height-1 prime $p\subset q$ such that A/p is factorial, because if there was such a p, the ideal q would be generated by 2 elements. However we will prove that:

Corollary 1. *Any height-2 prime $q\subset A$ contains a height-1 prime $p\subset q$ such that A/p is normal and the class group $C(A/p)$ is cyclicly generated by q/p.*

Analogously, given a smooth connected curve C in the complex projective space $\mathbb{P}^3$, there does not exist in general a smooth surface S containing C and having Picard number $\varrho(S)=1$, because if there was such an S, the curve C would be a complete intersection. However it will follow that:

Corollary 2. *Any smooth connected curve C in $\mathbb{P}^3$ lies on a smooth surface $S\subset\mathbb{P}^3$ with $\varrho(S)=2$ such that the Picard group* $\mathrm{Pic}(S)$ *is generated by C together with the hyperplane section.*

Both corollaries will be consequences of the following:

Theorem. *Let W be a smooth complex projective 3-fold and $B\subset W$ a reduced curve. Then there exists an irreducible normal surface $T\subset W$ containing B, with* $\mathrm{Sing}(T)\subset\mathrm{Sing}(B)$ *and such that one has an exact sequence*

$$0\to\mathrm{Pic}(W)\to C(T)\to\bigoplus_{i=1}^{m}\mathbb{Z}[B_i]\to 0$$

where $B_1,\dots,B_m$ are the irreducible components of B.

Note that the existence of a normal $T\subset B$ with $\mathrm{Sing}(T)\subset \mathrm{Sing}(B)$ is well known and is an easy consequence of the classical Bertini theorem (see [1;5]); the new information in the Theorem is that concerning the class group.

The above statements will be proved in Sections 1 and 2.

In Section 3 we make some variations on the Theorem and give further applications.

1. Proof of Corollary 2

In this section we assume the Theorem holds and we prove the corollaries. Corollary 2 is clear. To prove Corollary 1, suppose $A=S^{-1}A_1$ where A_1 is a finitely generated $\mathbb{C}$-algebra and $S\subset A_1$ is a multiplicative system. Put $X_1=\mathrm{Spec}(A_1)$, and $X_0=\mathrm{Reg}(X_1)$; then the image of $\mathrm{Spec}(A)=X\to X_1$ will be contained in X_0. Let $Y\subset X_0$ be the irreducible curve defined by q in X_0. By Hironaka's resolution of singularities, X_0 is contained as a Zariski open set in a smooth projective 3-fold W. Let B be the closure of Y in W. Now choose a surface $T\subset W$ as in the Theorem, denote by p the prime ideal in A corresponding to T and consider the surjective map

$$\mathrm{Pic}(W)\oplus\mathbb{Z}[B]\simeq C(T)\to C(A/p).$$

To conclude, it is sufficient to note that $\mathrm{Pic}(W)\to C(A/p)$ factorizes through $\mathrm{Pic}(X)=0$.

2. Proof of the Theorem

In this section we prove the Theorem.

By Hironaka's embedded resolution of singularities (but in fact the embedded resolution in [9, p. 218] will suffice) there exists a propermorphism $g:V\to W$ where V is a smooth projective 3-fold such that if we put $R=\mathrm{Sing}(B)$, $E=g^{-1}(R)$ and $C=$ strict transform of B, then E is a divisor, $C\to B$ is a normalization and $g:V\setminus E\to W\setminus R$ is an isomorphism. Let $C_1,\dots,C_m$ be the components of C ($C_i=$ normalization of B_i). Let $f:U\to V$ be the blowing up of V along C, $Z_i=f^{-1}(C_i)$, $Z=Z_1\cup\dots\cup Z_m$ and choose a very ample divisor H on V. Let $E_1,\dots,E_r$ be the irreducible components of E and $G_1,\dots,G_r$ their strict transforms on U. Put $\mathbb{P}^N=|nf^*H-Z|^V$.

We need the following:

Lemma. *$|nf^*H-Z|$ is very ample for $n\gg 0$.*

Proof. Note first that $|nf^*H-Z|$ is base point free for $n\gg 0$. Indeed this linear system certainly has no base points outside Z provided $n\gg 0$. Now if $x\in Z$ and $y=f(x)$, then x may be identified with a plane P contained in the Zariski tangent space T_yV and containing T_yC. Since C is smooth, there exists a germ of smooth surface around y, passing through C and not tangent to P. Finally by Serre's theorem on global generation applied to the twisted ideal sheaf of C, the above germ lifts to a global surface in $|nH|$ which passes through C and is not tangent to P at y; this surface will correspond to a member of $|nf^*H-Z|$ not passing through x.

Now if n_0 and n_1 are integers such that $\mathcal{O}_U(n_0f^*H-Z)$ and $\mathcal{O}_V(n_1H-K_V)$ are spanned by global sections then, $\mathcal{L}=\mathcal{O}_U(nf^*H-f^*K_V-3Z)$ is spanned by global sections provided $n \geq 3n_0+n_1$. Since the image of the corresponding morphism $U \to |\mathcal{L}|^V$ clearly has dimension 3 we get by the Grauert-Riemenschneider vanishing theorem [6], [11] that $H^2(U, \mathcal{L}^{-1})=0$, hence by Serre's duality $H^1(\mathcal{O}_U(nf^*H-2Z))=0$.

Let us show that $|nf^*H-Z|$ separates tangent vectors on U for $n \gg 0$ (in the same way one may prove it separates points and we will be done). Take $x \in U$ and $t \in T_xU$.

If $x \notin Z$, choose $D_0 \in |(n-1)f^*H-Z|$ with $x \notin D_0$ and choose $H_0 \in |H|$ with $y=f(x) \in H_0$ and $(T_xf)(t) \notin T_yH_0$. Then $D=f^*H_0+D_0$ contains x and is not tangent to t. If $x \in Z$ and $t \notin T_xZ$, choose $D_1 \in |nf^*H-2Z|$ with $x \notin D_1$. Then $D=D_1+Z$ contains x and is not tangent to t. Finally if $x \in Z$ and $t \in T_xZ$, consider the exact sequence

$$H^0(\mathcal{O}_U(nf^*H-Z)) \to \bigoplus_{i=1}^{m} H^0(\mathcal{O}_{Z_i}(nf^*H-Z_i)) \to H^1(\mathcal{O}_U(nf^*H-2Z))=0$$

and we are done because $\mathcal{O}_{Z_i}(nf^*H-Z_i)$ are very ample for $n \gg 0$ [7, p. 385].

Now returning to the proof of the Theorem, if F is a fibre of $Z \to C$ one gets $(F \cdot nf^*H-Z)=1$, hence in the embedding $U \subset \mathbb{P}^N$ the fibres F are straight lines. We claim there is a non-empty Zariski open set Λ_1 of hyperplanes in $\mathbb{P}^N$ meeting transversally every fibre F; indeed the set of hyperplanes in $\mathbb{P}^N$ containing a fibre is an $(N-2)$-plane in $\check{\mathbb{P}}^N$ ($=$dual space of $\mathbb{P}^N$) hence the set of hyperplanes containing at least a fibre of $Z \to C$ is an $(N-1)$-dimensional subset of $\check{\mathbb{P}}^N$. Let Λ_2 be a non-empty Zariski open set in $\check{\mathbb{P}}^N$ whose members have an irreducible intersection with each G_i. Now choose a member D of $|nf^*H-Z|$ which is generic (in Weil's sense) over thefield of definition of all varieties and morphisms appearing until now. Since $D \in \Lambda_1$, it is easy to see that the projection $D \to S=f(D) \in |nH|$ is an isomorphism. Now for $n \gg 0$ (if $W=V=\mathbb{P}^3$, then $n \geq 4$ will suffice) we have $h^{2,0}(D)=h^{2,0}(S)>h^{2,0}(V)=h^{2,0}(U)$. We will apply Noether's theorem in Lefschetz' form:

Theorem ([8] and [10, p. 264]). *If X is a smooth projective 3-fold, Λ is a very ample linear system on it and Y is a member of Λ which is generic in Weil's sense and if $h^{2,0}(Y)>h^{2,0}(X)$, then the restriction map* $\operatorname{Pic}(X) \to \operatorname{Pic}(Y)$ *is an isomorphism.*

In our case $\operatorname{Pic}(U) \to \operatorname{Pic}(D)$ is an isomorphism. Furthermore since $D \in \Lambda_2$, $S \cap E_i$

are irreducible. Clearly $T=g(S)$ is normal with $\mathrm{Sing}(T)\subset R$ and $S\setminus E\simeq T\setminus R$. The exact sequence

$$0\to \mathrm{Pic}(V)\to \mathrm{Pic}(U)\to \bigoplus_{i=1}^{m} \mathbb{Z}[Z_i]\to 0$$

together with the isomorphisms $\mathrm{Pic}(S)\simeq \mathrm{Pic}(D)\simeq \mathrm{Pic}(U)$ yield a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & & \\
\bigoplus_{j=1}^{r} \mathbb{Z}[E_j] \arrow[r] \arrow[d] & \mathrm{Pic}(V) \arrow[r] \arrow[d] & \mathrm{Pic}(W) \arrow[r] \arrow[d] & 0 \\
\bigoplus_{j=1}^{r} \mathbb{Z}[S\cap E_j] \arrow[r] \arrow[d] & \mathrm{Pic}(S) \arrow[r] \arrow[d] & C(T) \arrow[r] & 0 \\
0 & \bigoplus_{i=1}^{m} \mathbb{Z}[C_i] \arrow[d] & & \\
 & 0 & &
\end{tikzcd}$$

A diagram chase immediately yields the exact sequence appearing in the Theorem.

3. Applications

In this section we make some further discussion and give some more applications.

Proposition 1. *Let Λ be a (possibly non-complete) linear system on a smooth complex projective variety W, let $\varphi : W \dashrightarrow \mathbb{P}$ be the associated rational map and suppose that*

(1) $\dim \varphi(W)\geq 3$.

(2) *There exists a non-empty Zariski open subset Λ_1 in Λ all of whose members are irreducible and normal.*

Then there exists a non-empty Zariski open subset Λ_0 of Λ_1 such that for any $T\in\Lambda_0$ the cokernel of the map $\mathrm{Pic}(W)\to C(T)$ *is finitely generated.*

Proof. Let B be the base locus of Λ. By Hironaka's resolution of singularities, there exist morphisms $f: U \to W$ and $g: U \to \mathbb{P}$ such that $g = \varphi \circ f$, where U is smooth projective and $U \setminus f^{-1}(B) \to W \setminus B$ is an isomorphism. Now by Bertini's theorem there exists a Zariski open subset Λ_2 of hyperplanes H in $\mathbb{P}$ such that for $H \in \Lambda_2$, $S = g^*H$ is smooth and contains no component of $f^{-1}(B)$. Put $T = f(S)$ and $\Lambda_0 = \Lambda_1 \cap \Lambda_2$. Then for $H \in \Lambda_0$, $S \cap (U \setminus f^{-1}(B))$ is dense in S and isomorphic to $T \cap (W \setminus B)$; in particular S is irreducible. Since $\dim g(U) \geq 3$ and $\mathcal{O}_U(S)$ is spanned by global sections it follows by the Grauert–Riemenschneider vanishing theorem [6], [11] that $H^i(\mathcal{O}_U(-S)) = 0$ for $i = 1, 2$, so $\beta: H^1(\mathcal{O}_U) \to H^1(\mathcal{O}_S)$ is an isomorphism. But β is the tangent map of $\alpha: \mathrm{Pic}^0(U) \to \mathrm{Pic}^0(S)$ at the origin of $\mathrm{Pic}^0(U)$ so α is an isogeny, in particular it is surjective. By the Neron–Severi theorem we get that $\mathrm{coker}(\mathrm{Pic}(U) \to \mathrm{Pic}(S))$ is finitely generated. Now $\mathrm{coker}(\mathrm{Pic}(W) \to C(T \setminus B))$ being a quotient of the previous coker is also finitely generated. Finally $C(T)$ is finitely generated because it is an extension of $C(T \setminus B)$ by some finitely generated group and we are done.

To apply Proposition 1 consider a smooth complex projective 3-fold W, a very ample divisor H on W and a closed subscheme B of dimension 1 in W. Let $|nH - B|$ be the largest linear subsystem of $|nH|$ having scheme-theoretic base locus B and put $\Sigma = \{x \in B, \dim T_{x,B} \geq 3\}$. The following is an easy consequence of Bertini type results in [1,5]:

Remark. The conditions below are equivalent:

(1) B is contained in a normal surface $T \subset W$.

(2) $\dim \Sigma = 0$.

(3) For $n \gg 0$ there exists a non-empty Zariski open subset of $|nH - B|$ all of whose members are normal.

So if $\dim \Sigma = 0$ and $n \gg 0$, the above Proposition 1 applies to $\Lambda = |nH - B|$ and we can ask what is the generic rank of $\mathrm{coker}(\mathrm{Pic}(W) \to C(T))$ when T runs through the set of normal members of $|nH - B|$. By Proposition 1 this rank is finite and the proof of the Theorem shows for instance that

$$\text{generic rank} = \text{number of components of } B$$

provided B is a smooth curve. Genericity is understood here in Weil's sense.

In the end we will show how using the same type of arguments as in Sections 1 and 2 (and in fact easier ones) one may compute the minimum rank of class groups for members of non-complete linear systems in 3-space having zero-dimensional base locus. Here is an 'affine' application (A being as in Corollary 1):

Proposition 2. *Let $r \geq 2$ be a natural number and $M \subset A$ a height-3 prime ideal. Then there exists a prime element $x \in M^r \setminus M^{r+1}$ such that A/xA is factorial. Suppose furthermore $A = \mathbb{C}[t_1, t_2, t_3]$ and L_{rs} is the $\mathbb{C}$-vector space of all polynomials of the form*

$$F = F_r + F_{r+1} + \cdots + F_{r+s}$$

where $s \geq 0$ *and* F_j *is homogeneous of degree* j. *If* F *is generic in* L_{rs} *(in Weil's sense), then*

$$C(A/FA) = \begin{cases} 0 & \text{if } s \geq 2, \\ \mathbb{Z}^{r^2+r-1} & \text{if } s = 1, \\ J \oplus (\mathbb{Z}/r\mathbb{Z}) & \text{if } s = 0 \end{cases}$$

where J *is the Jacobian of the curve* $\mathrm{Proj}(A/FA)$.

The above statement about $\mathbb{C}[t_1, t_2, t_3]$ is a very special case of a problem raised by Dolgachev [3]. Case $s = 0$ is of course well known. Case $s = 1$ is also classical.

Proof. Choose a smooth projective variety W and morphisms $\mathrm{Spec}(A) \to \mathrm{Reg}(\mathrm{Spec}(A_1)) \subset W$ as in Section 1 (if $A = \mathbb{C}[t_1, t_2, t_3]$, take

$$W = \mathbb{P}^3 = \mathrm{Proj}(\mathbb{C}[t_0, t_1, t_2, t_3])).$$

Choose a very ample divisor H on W (if $W = \mathbb{P}^3$, take $H \in |\mathcal{O}(1)|$) and let $f: U \to W$ be the blowing up of W at the point y corresponding to M. Denote by $Y \subset U$ the exceptional plane. It is easy to see that $|f^*H - Y|$ is base point free and that there exists an integer n_0 such that $|nf^*H - Y|$ is very ample on U provided $n \geq n_0$, hence for any $r \geq 1$ and $n \geq n_0 + r - 1$, $|nf^*H - rY|$ is very ample (for $(W, H) = (\mathbb{P}^3, \mathcal{O}(1))$ it is sufficient to take $n_0 = 2$). Choose as in Section 2 a Weil generic member $S \in |nf^*H - Y|$, $n \geq n_0 + r - 1$. By genericity $S \cap Y$ must be irreducible. Now we have an exact sequence

$$0 \to H^2(\mathcal{O}_U) \to H^2(\mathcal{O}_S) \to H^3(\mathcal{O}_U(-S)) \to H^3(\mathcal{O}_U) \to 0$$

and we also have

$$\dim(H^3(\mathcal{O}_U(-S))) = \dim(H^0(\mathcal{O}_U(f^*(K_W + nH) - (r-2)Y)))$$

which obviously goes to ∞ when $n \to \infty$. Put $T = f(S)$: since $S \cap Y$ has degree r in Y, T has multiplicity r in y. Now exactly as in Section 2 we get an exact sequence

$$0 \to \mathrm{Pic}(W) \xrightarrow{\alpha} C(T) \to G \to 0$$

where $G = \mathrm{coker}(\mathrm{Pic}(U) \to \mathrm{Pic}(S))$. Recall that by Lefschetz's theory [2], G is finitely generated and torsion free. If $n \gg 0$ we have $h^{2,0}(S) > h^{2,0}(U)$ (for $(W, H) = (\mathbb{P}^3, \mathcal{O}(1))$ it is sufficient to take $n \geq r + 2$), hence $G = 0$. If x is the prime element in A corresponding to T, the surjection $C(T) \to C(A/xA)$ yields exactly as in Section 1 facoriality of A/xA.

Now suppose $A = \mathbb{C}[t_1, t_2, t_3]$. The case when F is homogenous being well known, we suppose $s \geq 1$. Since S is generic, the intersection $T \cap P$ between T and the hyperplane P given by the equation $t_0 = 0$ is irreducible, so we have an exact sequence

$$\mathbb{Z}[T\cap P]\xrightarrow{\beta} C(T)\to C(A/FA)\to 0.$$

Since $\mathrm{Im}(\alpha)=\mathrm{Im}(\beta)$, we get $C(A/FA)=G$, so $C(A/FA)=\mathbb{Z}^{\varrho-2}$ where $\varrho=\varrho(S)$.

We have already seen that $\varrho=2$ for $n\geq r+2$. Now for $n=r+1$, T has degree $r+1$ and an r-tuple point. Such surfaces are very classical beings. Projection from the r-tuple point gives their rationality. Finally the adjunction formula gives $(K_S^2)=-r^2-r+9$, so $\varrho(S)=10-(K_S^2)=r^2+r+1$ and we are done.

Final Remark. Most of what we did in this paper for curves, surfaces and 3-folds holds in the higher-dimensional case with essentially the same proofs. One has to replace

"3-fold" by "manifold of dimension ≥ 3",

"surface" by "hypersurface",

"curve" by "reduced subscheme of pure codimension 2".

Then the statement of Corollary 1 remains unaltered; however in Corollary 2 and in the Theorem the conclusions about the singular locus of the hypersurface will fail.

Note. The case $s\geq 2$ in our Proposition 2 from Section 3 was previously obtained by J. Kollàr (1982, unpublished) as we found out after submitting the manuscript. J. Kollàr also proved the existence of certain factorial rational double points using moduli of $K3$ surfaces.

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